Linear Algebra

[KOMS120301] - 2023/2024

6.2 - Inverses and its relation to the Gaussian method, Gauss-Jordan method, and linear system

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Learning objectives

After this lecture, you should be able to:

1. find inverse by Gaussian elimination algorithm;
2. find inverse by Gauss-Jordan elimination algorithm;
3. explain the method to find solution of linear system using inverse;
4. finding solution of a linear system using inverse;
5. solving homogeneous system (when the constant vector is a zero vector).

Part 1: Algorithms to find an inverse

Algorithm

- Computing inverse by Gaussian elimination

  Given an **invertible square matrix** A. To compute A^{-1}, we perform the following computation:

$$[A \mid I] \xrightarrow{\text{G-J elimination}} [I \mid A^{-1}]$$

- Computing inverse by Gauss-Jordan elimination

  Given an **invertible square matrix** A. To compute A^{-1}, we perform the following computation:

$$[A \mid I] \xrightarrow{\text{Gaussian elimination}} [I \mid A^{-1}]$$

Example 1

Find the inverse of: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

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Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \underset{R3 - R1}{\overset{R2 - 2R1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \overset{R3 + 2R2}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \overset{R3/(-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3- & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \overset{R1 - 2R2}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \overset{R1 - 2R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] = [I \mid A^{-1}]$$

Example 1 (*cont.*)

Hence, $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

It can be checked that:

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2

Apply G-J method to find the inverse of: $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \underset{R3+R1}{\overset{R2-2R1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \overset{R2/(-8)}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 9/8 & 2/8 & -1/8 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \overset{R3-8R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 9/8 & 2/8 & -1/8 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \overset{R2/(-8)}{\sim}$$

The reduced form contains a **zero row** (hence, there is no way to create an identity matrix on the left block).

This means that A has no inverse.

Example 2 (*cont.*)

It can be checked that A has **zero determinant**.

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{vmatrix} \\ &= 1(4)(5) + 6(-1)(-1) + 4(2)(2) - 4(4)(-1) - (-1)(1)(2) - 5(6)(2) \\ &= 20 + 6 + 16 + 16 + 2 - 60 \\ &= 0\end{aligned}$$

Exercise

If exist, determine the inverses of the following matrices!

- $$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
- $$\begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$$
- $$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$
- $$\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$$

Exercise

Solve the following linear system using Gauss-Jordan elimination:

- $\begin{cases} a - b + 2c - d = -1 \\ 2a + b - 2c - 2d = -2 \\ -a + 2b - 4c + d = 1 \\ 3a - 3d = -3 \end{cases}$

- $\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$

Part 3: Relation to Linear System

Relation to linear system

Recall that the system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

can be written as matrix operation: $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix, $\mathbf{x}$ is the variable vector, and $\mathbf{b}$ is the constant matrix.

- If A is invertible, then the system has a unique solution;
- Otherwise, the solution is not unique.

Algorithm

Suppose we want to solve: $A\mathbf{x} = \mathbf{b}$, where $\det(A) \neq 0$.

Multiplying both sides with A^-1 (from left), we obtain:

$$
\begin{aligned}
(A^{-1})\ A\mathbf{x} &= (A^{-1})\ \mathbf{x} \\
I\mathbf{x} &= A^{-1}\ \mathbf{b} \quad \text{since } AA^{-1} = I \\
\mathbf{x} &= A^{-1}\ \mathbf{b} \quad \text{since } I\mathbf{x} = \mathbf{x}
\end{aligned}
$$

Hence, the solution of the system $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

Example: finding solution of linear system using inverse

Given a linear system:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ x_1 \qquad\quad + 8x_3 = 1 \end{cases}$$

Solution:

We have compute the inverse of: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, that is,

$A = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$. Hence, the solution is:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

You should be able to check that **x** matches with the solution obtained using Gaussian or Gauss-Jordan elimination.

Homogeneous case

If the system is homogeneous (i.e., $\mathbf{b} = \mathbf{0}$), then the following hold:

- If A is invertible, then the system only has the trivial solution;
- If A is not invertible, then the system has non-trivial solution.

Example of homogeneous system

Show that the following homogeneous system only has the trivial solution!

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ x_1 \qquad\quad + 8x_3 = 0 \end{cases}$$

Show that the following homogeneous system has a non-trivial solution!

$$\begin{cases} x_1 + 6x_2 + 4x_3 = 0 \\ 2x_1 + 4x_2 - x_3 = 0 \\ -x_1 + 2x_2 + 5x_3 = 0 \end{cases}$$

Example of homogeneous system (*cont.*)

First example:

The homogeneous system has coefficient matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ and

$\det(A) \neq 0$ with $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Second example:

The homogeneous system has coefficient matrix: $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

It can be verified that $\det(A) = 0$, so A^{-1} does not exist.

The system has a non-trivial solution, for instance:

$$x_1 = -29,\ x_2 = 8,\ x_3 = -9$$

Advantage of using inverse-method in solving linear system

Inverse-method is useful to solve linear system $A\mathbf{x} = \mathbf{b}$ with the same coefficient matrix A but with different constant vector $\mathbf{b}$.

For example:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ x_1 \qquad\quad + 8x_3 = 1 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 3x_3 = 10 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ x_1 \qquad\quad + 8x_3 = -2 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 3x_3 = -4 \\ 2x_1 + 5x_2 + 3x_3 = 12 \\ x_1 \qquad\quad + 8x_3 = 5 \end{cases}$$

Can you explain why?

- Since $\mathbf{x} = A^{-1}\mathbf{b}$, then to solve those systems, it is enough to compute A^{-1} **once**, then multiply it with the corresponding vector $\mathbf{b}$.

Exercise 1

Solve the following system using inverse-method:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ x_1 \qquad\quad + 8x_3 = 1 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 3x_3 = 10 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ x_1 \qquad\quad + 8x_3 = -2 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 3x_3 = -4 \\ 2x_1 + 5x_2 + 3x_3 = 12 \\ x_1 \qquad\quad + 8x_3 = 5 \end{cases}$$

Exercise 2

Solve the following linear system using inverse-method:

- $$\begin{cases} a - b + 2c - d = -1 \\ 2a + b - 2c - 2d = -2 \\ -a + 2b - 4c + d = 1 \\ 3a - 3d = -3 \end{cases}$$

- $$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

to be continued...